

# Spontaneous Symmetry Breaking at Infinite Momentum without $P^+$ Zero-Modes

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## Abstract

The nonrelativistic interpretation of quantum field theory achieved by quantization in an infinite momentum frame is spoiled by the inclusion of a mode of the field carrying  $p^+ = 0$ . We therefore explore the viability of doing without such a mode in the context of spontaneous symmetry breaking (SSB), where its presence would seem to be most needed. We show that the physics of SSB in scalar quantum field theory in  $1 + 1$  space-time dimensions is accurately described without a zero-mode.

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The infinite momentum frame provides a vehicle for casting any relativistic quantum mechanical system in terms of the (nonrelativistic) quantum dynamics of Heisenberg and Schrödinger inspired by the classical dynamics of Galileo and Newton [1]. This Newtonian view of quantum field theories [2, 3] might arguably be dismissed as a mere curiosity, since those theories have several satisfactory manifestly relativistic formulations. But the corresponding view of string theory [4] remains one of the best hopes for a truly fundamental description of string that does not rely on perturbation theory [5]. Thus it behooves us to probe the viability of the Newtonian view of quantum field theory [6], since the latter might well be merely a low energy effective theory for string.

In quantum field theory the best way to achieve the nonrelativistic description is to employ light front coordinates [1]  $x^\pm = (x^0 \pm x^3)/\sqrt{2}$ , choosing  $x^+$  as time, and referring the  $x^-$  coordinate to its conjugate momentum labelled by  $p^+ > 0$ . Then  $p^+$  assumes the role of a variable Newtonian mass. A typical quantum field, for instance a real scalar field  $\phi$ , has the expansion

$$\phi(\mathbf{x}, x^-, x^+) = \int_0^\infty \frac{dp^+}{\sqrt{4\pi p^+}} \left( a(\mathbf{x}, p^+) e^{-ix^- p^+} + a^\dagger(\mathbf{x}, p^+) e^{+ix^- p^+} \right), \quad (1)$$

where the quantum nature of the field is fixed by imposing

$$\begin{aligned} [a(\mathbf{x}, p^+), a^\dagger(\mathbf{y}, q^+)] &= \delta(\mathbf{x} - \mathbf{y}) \delta(p^+ - q^+) \\ [a(\mathbf{x}, p^+), a(\mathbf{y}, q^+)] &= 0. \end{aligned} \quad (2)$$

Discretizing  $p^+ = lm$ ,  $l = 1, 2, \dots$ , sharpens the Newtonian interpretation, because then each value of  $l$  labels a different species of particle with Newtonian mass  $lm$  [6]. However, much debate and controversy in the discretized light-cone quantization (DLCQ) literature (for a review see [7]) has centered on the possible necessity of including a field mode carrying  $p^+ = 0$  [8]. It has been suggested that without such a mode it would be difficult if not impossible to describe such a commonplace field theoretic phenomenon as spontaneous symmetry breaking [9]. If a zero-mode were really necessary, we would have no good Newtonian interpretation for it, and the field theory would not be adequately described by Newtonian dynamics.

While conceding that the inclusion of a zero-mode is a valid field theoretic option, we argue in this letter that it is not necessary, even to describe spontaneous symmetry breaking. First of all, the physics of condensation associated with SSB does not require a fundamental zero-mode. Just consider the Cooper pairs of BCS superconductivity, which most definitely carry Newtonian mass. Similarly, in the infinite momentum frame there is no compelling reason to require that a condensate carry zero  $p^+$ . It is only necessary that in the infinite volume limit, local physics cannot extract  $p^+$  from or deposit  $p^+$  into the condensate.

In this letter, we shall give a detailed analysis of the simplest field theory that exhibits SSB, namely a real scalar field in 1 space dimension. At the outset, our quantum field will have no zero-mode. We shall show that, in spite of this, the physics of SSB is completely and accurately described in our model.

The light-cone Hamiltonian is  $P^-$ , the density of which we choose to be

$$\mathcal{H} = -\frac{\mu^2}{2} : \phi^2 : + \frac{\lambda}{24} : \phi^4 : . \quad (3)$$

With discrete  $p^+ = lm$  (equivalently periodic boundary conditions in  $x^-$  on the interval  $-\pi/m < x^- < \pi/m$ ), the field has the expansion

$$\phi = \sum_{l=1}^{\infty} \frac{1}{\sqrt{4\pi l}} \left[ a_l e^{il\theta} + a_l^\dagger e^{-il\theta} \right], \quad (4)$$

where we have defined the angle  $\theta = -mx^-$ . The quantum conditions are then simply  $[a_j, a_l^\dagger] = \delta_{jl}$ ,  $[a_j, a_l] = 0$ . Note the complete absence of a zero-mode. It is convenient to also define a rescaled Hamiltonian,  $h \equiv mH/\mu^2$ ,

$$h = -\sum_{l>0} \frac{a_l^\dagger a_l}{2l} + \frac{g}{4} \sum_{l_1+l_2>l_3>0} \frac{a_{l_1+l_2-l_3}^\dagger a_{l_3}^\dagger a_{l_1} a_{l_2}}{\sqrt{l_1 l_2 l_3 (l_1 + l_2 - l_3)}} + \frac{g}{6} \sum_{l_1, l_2, l_3 > 0} \frac{a_{l_1+l_2+l_3}^\dagger a_{l_3} a_{l_1} a_{l_2} + a_{l_1}^\dagger a_{l_2}^\dagger a_{l_3}^\dagger a_{l_1+l_2+l_3}}{\sqrt{l_1 l_2 l_3 (l_1 + l_2 + l_3)}}, \quad (5)$$

where  $g \equiv \lambda/8\pi\mu^2$ . The dynamical system has now been completely specified. The negative quadratic term is designed to drive the instability towards spontaneous symmetry breaking. The Hamiltonian possesses a parity symmetry under  $\phi \rightarrow -\phi$ , and also conserves discrete total  $P^+ = Mm$ . A brute force way to analyze the dynamics would be to look for energy eigenstates with definite  $P^+$ , that is fixed  $M$ . The state space in this subspace has dimension  $p(M)$ , the number of unordered partitions of the integer  $M$ . As long as  $M$  is not too large ( $p(M)$  increases exponentially with  $\sqrt{M}$ ), the Hamiltonian can be numerically diagonalized with the aid of a computer (results of our analysis for  $M \leq 17$  can be seen in Fig. 1 and Fig. 2).

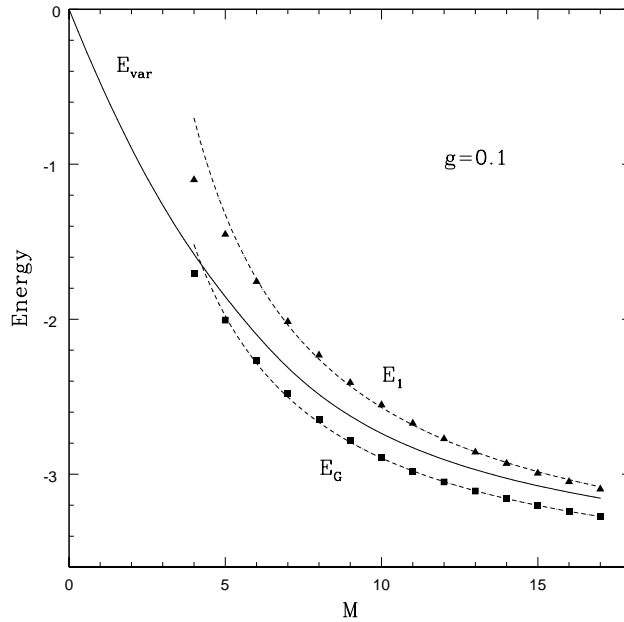


Figure 1: Energy as a function of  $M$  of the ground state ( $E_G$ ) and the first excited state ( $E_1$ ) compared to the variational calculation (solid curve) – for  $g = 0.1$ . The dashed curves are fits to the eigenvalues in the range  $10 \leq M \leq 17$  as described later in the text.

However, actual symmetry breaking can occur only in the infinite volume limit. This is taken by letting  $m \rightarrow 0$  and  $M \rightarrow \infty$  keeping  $P^+ = Mm$  fixed. Symmetry breaking occurs when the lowest energy level is degenerate, possessing two states of opposite parity. But at finite volume, tunneling between these states always lifts the degeneracy. Thus all one can hope to find in the brute force numerical method is a gradual approach to degeneracy as  $M$  is increased. Our initial

studies in this direction were equivocal: we could not deal with sufficiently large  $M$  to definitively reveal such a trend.

Since the model is expected to show symmetry breaking for arbitrarily weak coupling, we should be able to confirm this analytically. Since the weak coupling limit is semi-classical, it is natural to apply the variational principle, choosing as a trial the coherent state<sup>‡</sup>  $|\alpha\rangle = e^{\sum \alpha_l a_l^\dagger} |0\rangle e^{-\sum \alpha_l a_l^*/2}$ . The energy functional is then

$$\langle \alpha | h | \alpha \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left[ -\frac{1}{4} f^2 + \frac{g}{24} f^4 \right], \quad (6)$$

with

$$f(\theta) = \sum_{l=1}^{\infty} \frac{1}{\sqrt{l}} \left[ \alpha_l e^{il\theta} + \alpha_l^* e^{-il\theta} \right]. \quad (7)$$

The minimum occurs when  $f^2(\theta) = 3/g$ . Since there is no zero-mode, the simplest solution of this condition is given by

$$f(\theta) = \begin{cases} +\sqrt{3/g} & \text{for } 0 \leq |\theta| < \pi/2 \\ -\sqrt{3/g} & \text{for } \pi/2 < |\theta| < \pi. \end{cases} \quad (8)$$

Then the  $\alpha$ 's are determined to be

$$\alpha_{2n} = 0, \quad \alpha_{2n+1} = \frac{2}{\pi} \sqrt{\frac{3}{g}} \frac{(-)^n}{\sqrt{2n+1}}. \quad (9)$$

Because of the discontinuities, the expectation value of  $P^+/m = \sum_l l a_l^\dagger a_l$  in this trial state is infinite:  $\sum l |\alpha_l|^2 = (12/\pi^2 g) \sum_n 1$ .

There are two ways to extend this variational approach to the situation of finite  $M$ . One is to work with the projection of the state  $|\alpha\rangle$  to the subspace with definite  $M$ :  $|\alpha, M\rangle = P_M |\alpha\rangle$ . This is easy enough for specific  $M$  values but difficult to do for general  $M$ . A more tractable approach is to do a constrained variation: minimizing  $H$  subject to the constraint  $\langle P^+ \rangle = mM$ , with  $M$  the number of  $P^+$  units. This can be done by adding a Lagrange multiplier term  $\beta(P^+/m - M)$  to  $h$  and minimizing the expectation of the resulting operator.  $\beta$  can then be adjusted so that the constraint is satisfied. Instead of  $\langle \alpha | h | \alpha \rangle$ , we now minimize

$$\langle \alpha | h_\beta | \alpha \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left[ \beta \left( \frac{f'^2}{2} - M \right) - \frac{1}{4} f^2 + \frac{g}{24} f^4 \right], \quad (10)$$

again with the understanding that  $f(\theta)$  has no zero-mode and has period  $2\pi$ . Denoting the maximum of  $f$  by  $f_0$ , the simplest solution of all these conditions is

$$f(\theta) = f_0 \operatorname{sn} \left( \frac{2\theta + \pi}{\pi} K, k \right), \quad g f_0^2 = \frac{6k^2}{1+k^2}, \quad (11)$$

where  $k$  is the modulus of the elliptic function, following the conventions of [11]. Note that we have located  $\theta = 0$  at a maximum of the sn function.  $K(k)$  is the quarter period of sn given by the complete elliptic integral of the first kind

$$K = K(k) = \int_0^1 dt (1-t^2)^{-1/2} (1-k^2 t^2)^{-1/2}, \quad (12)$$

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<sup>‡</sup>A coherent state involving only zero-modes was used in an earlier attempt to understand the vacuum of this field theory [10].

and  $\beta$  is related to  $k^2$  by  $\beta = \pi^2/8(1+k^2)K^2$ .

The constraint  $\langle P^+ \rangle = mM$  links the last independent parameter to  $M$ :

$$\begin{aligned} M &= \frac{12k^2}{\pi^2 g(1+k^2)^{3/2}} K \int_0^1 dt \sqrt{1-t^2} \sqrt{1-k^2 t^2} \\ &= \frac{4K}{\pi^2 g \sqrt{1+k^2}} \left( E - \frac{1-k^2}{1+k^2} K \right), \end{aligned} \quad (13)$$

where  $E = E(k) = \int_0^1 dt \sqrt{1-k^2 t^2} / \sqrt{1-t^2}$  is the complete elliptic integral of the second kind. We immediately see, for example, that the limit  $M \rightarrow \infty$  is achieved by taking  $k^2 \rightarrow 1$ . Indeed in this limit  $K \sim (1/2) \ln(16/(1-k^2))$ , so the approach of  $k^2$  to 1 is exponential  $k^2 \sim 1 - 16e^{-M\pi^2 g/\sqrt{2}}$ .

We finally come to the evaluation of the expectation of  $h$  in this trial state:

$$\langle \alpha | h | \alpha \rangle = -\frac{3k^2}{2g(1+k^2)^2} + \frac{2}{Mg^2\pi^2(1+k^2)} \left( E - \frac{1-k^2}{1+k^2} K \right)^2. \quad (14)$$

The large  $M$  behavior is easy to read off because  $k^2 \rightarrow 1$  exponentially in  $M$ . Thus to any finite order in  $1/M$  we merely set  $k^2 = 1$  in this expression

$$\langle \alpha | h | \alpha \rangle \sim -\frac{3}{8g} + \frac{2}{g^2\pi^2} \frac{1}{2M} + O\left(e^{-gM\pi^2/\sqrt{2}}\right). \quad (15)$$

The coefficient of  $1/2M$  in this expression is just the square of the infinite volume Lorentz invariant mass of the state in units of  $\mu^2$ . Since the state has a soliton anti-soliton pair this mass should be twice the soliton mass, so the calculation confirms that our setup leads to the correct soliton mass,  $M_{sol} = \mu/\pi g\sqrt{2} = 4\sqrt{2}\mu^3/\lambda$ .

We should stress that though our variational estimate for the energy only gives an upper bound on the true ground state energy at general coupling, it should actually approach the exact answer at weak coupling  $g \ll 1$ . (Indeed, Fig. 2 shows that for  $M = 16$  the variational estimate is quite good up to  $g = 0.5$ .) This is because our choice of trial function reduces the variational problem to that of the classical limit, which is equivalent to weak coupling. More precisely, we can say that the exact ground state energy should have the large  $M$  expansion

$$E_G \sim -\frac{3}{8g}(1 + O(g)) + \frac{1 + O(g)}{g^2\pi^2 M} + O\left(\frac{1}{M^2}\right). \quad (16)$$

Note that both of the exhibited terms dominate the corrections provided  $1/g \ll M \ll 1/g^2$ . In particular, the exact and variational energies are expected to have large  $M$  limits that differ by an amount of  $O(1)$  as  $g \rightarrow 0$ . This tendency is evident in Fig. 1. More quantitatively, fits to the data in Fig. 1 for the range  $10 \leq M \leq 17$  give:  $E_G = -3.81 + 18.34/2M$ ,  $E_{gap} = E_1 - E_G = 6.53/2M$ . The variational estimate of the ground state, Eq. 15, is  $-3.75 + 20.26/2M$  for  $g = 0.1$ , within 10% of the numerical fit. A corresponding estimate for the gap is less direct. Fluctuations about the trial coherent state are controlled by a  $\theta$  dependent mass squared  $(gf^2(\theta) - 1)\mu^2$  which approaches  $2\mu^2$  as  $gM \rightarrow \infty$  for almost all  $\theta$ . This infinite volume value yields an estimate  $2ME_{gap} \approx (4/g\pi) \approx 12.73$  for  $g = 0.1$ , nearly a factor of 2 larger than our fit. Unfortunately, the values of  $gM$  used in our fit were in the range 1 to 1.7, for which  $\int d\theta (gf^2 - 1)/2\pi$  varies from 1.14 to 1.49, indicating that  $12.73/2M$  is an overestimate for the gap. In contrast, the asymptotic form of the variational energy Eq.15 is quite accurate for  $gM > 1$  due to the  $\pi^2/\sqrt{2}$  in the exponential.

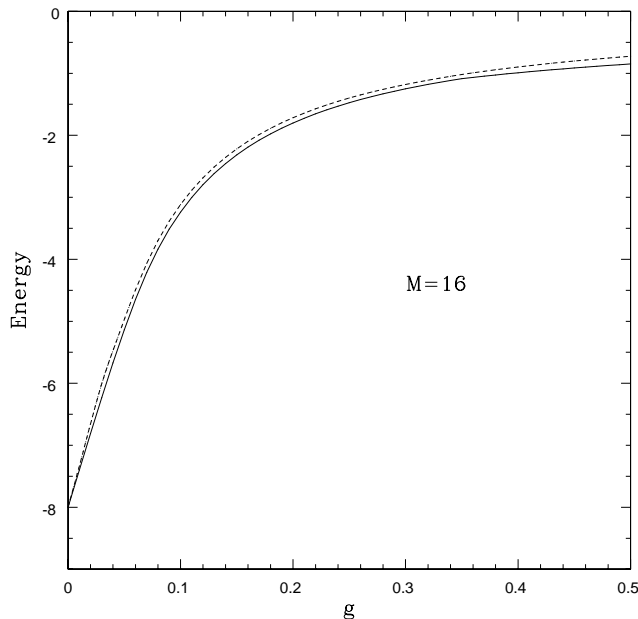


Figure 2: The energy dependence on  $g$  (for  $M = 16$ ) of the variational calculation (dashed curve) compared to the numerical calculation for the lowest eigenstate (solid curve).

The weak coupling limit also assures that the distribution of  $M$  values in the state  $|\alpha\rangle$  is sharply peaked about its mean value  $\langle P^+/m \rangle$ . Indeed a simple evaluation yields at large  $gM$ :  $\Delta M/M \approx \sqrt{3g}/\pi$  (where  $\Delta M = \sqrt{\langle (P^+/m)^2 \rangle - M^2}$ ). Thus it is in the weak coupling limit that the constrained variational approach is guaranteed to be equivalent to the projection onto a state of definite  $M$ . Moreover, in Fig. 3 we see that for  $M = 16$  they match quite well for  $g \leq 0.5$ .

We note in passing that the weak-coupling validity of the variational calculation also holds when  $M$  stays finite (although this is not so interesting for SSB). For example, if we examine  $g \rightarrow 0$  at fixed  $M$ , we find that the parameter  $k^2 \sim 2gM/3$ , so that  $\langle h \rangle \sim -M/2 + O(g)$ , indeed tending to the minimum eigenvalue of  $h$  at  $g = 0$ .

We can infer the values for the  $\alpha_l$  by developing  $f(\theta)$  in a Fourier series:

$$f(\theta) = \frac{2\pi}{K} \sqrt{\frac{6/g}{1+k^2}} \sum_{n=0}^{\infty} \frac{(-)^n q^{n+1/2}}{1-q^{2n+1}} \cos(2n+1)\theta, \quad (17)$$

where  $q \equiv \exp\{-\pi K(1-k^2)/K(k^2)\}$ . Thus we read off

$$\alpha_{2n} = 0, \quad \frac{\alpha_{2n+1}}{\sqrt{2n+1}} = \frac{\pi}{K} \sqrt{\frac{6/g}{1+k^2}} \frac{(-)^n q^{n+1/2}}{1-q^{2n+1}}. \quad (18)$$

Notice that for  $k^2$  near unity,  $q \sim 1$ ,  $(1-q^{2n+1})K \sim \pi^2(2n+1)/2$ , so that the  $\alpha$ 's revert to their step function values, which is to be expected since this limit corresponds to  $M \rightarrow \infty$ . On the other hand for  $k^2$  near 0, corresponding to weak coupling at fixed  $M$ , we see that  $q \rightarrow 0$ , so that the  $\alpha_1$  mode dominates. In other words, at weak coupling and fixed  $M$  the lowest energy state is obtained by putting all particles in the first mode, which can also be seen by simple inspection of the  $g = 0$  Hamiltonian.

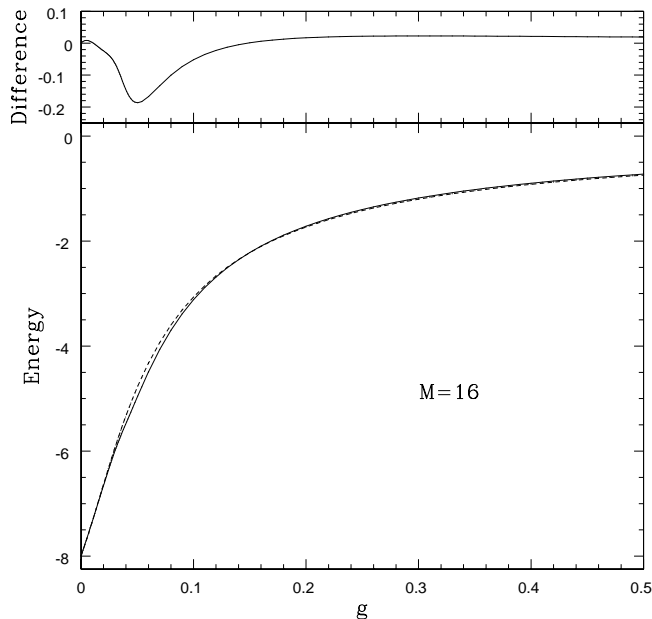


Figure 3: Comparison of the two variational approaches for finite  $M$ : Lagrange multiplier (solid curve) and simple projection (dashed curve) – for  $M = 16$ . The difference between these curves is presented on a separate scale.

Finally, we address the question of spontaneous symmetry breaking. The trial state  $|\alpha\rangle$  transforms to  $|\alpha\rangle$  under the discrete parity transformation, and so it is not invariant under the symmetry. Clearly the states  $|\pm\alpha\rangle$  have the same variational energy. However, one can form parity eigenstates  $|\pm\rangle = C(|\alpha\rangle \pm |-\alpha\rangle)$  which do not necessarily have the same variational energy:

$$\langle \pm | h | \pm \rangle = \frac{\langle \alpha | h | \alpha \rangle \pm \text{Re} \langle -\alpha | h | \alpha \rangle}{1 \pm \text{Re} \langle -\alpha | \alpha \rangle}. \quad (19)$$

The parity eigenstates also have slightly different mean  $P^+/m$  values:

$$\langle \pm | \frac{P^+}{m} | \pm \rangle = M \frac{1 \mp \text{Re} \langle -\alpha | \alpha \rangle}{1 \pm \text{Re} \langle -\alpha | \alpha \rangle}, \quad (20)$$

and one must take care to compare energies at the *same*  $\langle P^+ \rangle$ . However for large  $M$ , the  $M$  dependence of  $E$  is already suppressed, and thus this subtlety can be ignored. One can show that for large  $M$  the overlap  $\langle -\alpha | \alpha \rangle \sim (Mg\sqrt{2})^{-12/\pi^2 g}$ . This shows that at finite  $M$  (finite volume) the states  $|\pm\rangle$  are not quite degenerate, and the lower one does not break the symmetry. However, in the infinite volume limit  $M \rightarrow \infty$ ,  $\langle -\alpha | \alpha \rangle \rightarrow 0$  and degeneracy between opposite parity trial states is achieved, signalling SSB. Note that the splitting must vanish faster than  $1/M$ , because the energy scale of the infinite volume theory is set by  $1/M$  (recall that the true Hamiltonian is  $(\mu^2/m)h$  and  $Mm = P^+$  is fixed in the infinite volume limit). Within the variational approximation this puts an upper bound on the coupling for SSB, namely  $g < 12/\pi^2$ . Of course, the variational solution is guaranteed to be exact only as  $g \rightarrow 0$ , so that the precise value of this upper limit should be treated with caution.

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